Unit 2 - 1.1 Polynomials

Polynomial:

An expression of the form:

$a_nx^n + a_{n-1}x^{n-1} + \ldots\ldots\ldots\; a_3x^3 + a_2x^2 + a_1x + a_0$

with $a_0, \ldots.\; a_n$ constants and $a_n \neq 0$
is called a polynomial of degree n
(the highest power of x is the degree)

Division – some terms:

Divisor – what you are dividing by

Dividend – the number you are dividing into

Quotient – how many times the divisor goes into the dividend

Remainder – What is left over.

Quotient r. Remainder

Divisor | Dividend

Division of polynomials – Nested or synthetic division

Dividing f(x) by x – h

Note: the divisor **MUST** be in the form x – h

Example:

Find the quotient and remainder when $x^3 + 6x^2 + 3x – 15$ is divided by x - 3

Follow the working opposite.

The quotient is $\mathbf{x^2 + 9x + 30}$ and the **remainder is 75**

It should also be noted that the remainder when the divisor is x – 3 is f(3)

To illustrate this $\mathbf{f(3) = 3^3 + 6(3)^2 + 3(3) – 15 = 27 + 54 + 9 – 15 = 75}$

If we divided the quadratic f(x) by x – h then the remainder would be f(h)

Example of synthetic (nested) division:

Find the quotient and remainder when $x^3 + 6x^2 + 3x – 15$ is divided by x - 3

Write down the coefficients of the polynomial
– taking care to put a 0 where a power of x is missing

	A	B	C	D	
3	1	6	3	-15	1
	↓	3	27	90	2
	1	9	30	75	3

The shaded row and column are used there only to refer to the table for explanation.

Step 1. Put the contents of A1 straight down to A3
Step 2. Multiply A3 by the divisor and put result in B2
Step 3. Add B1 to B2 and put result in B3

Step 4. Multiply B3 by the divisor and put result in C2
Step 5. Add C1 to C2 and put result into C3

Step 6. Multiply C3 by divisor and put result into D2
Step 7. Add D1 and D2 and put result into D3

A3, B3, C3 are the coefficients of the quotient and D3 is the remainder.

Example:

Find the quotient and remainder when $x^3 + 6x^2 + 3x – 15$ is divided by x + 3

Note we **must** make x + 3 into **x – (-3)**

The **quotient is** $\mathbf{x^2 + 3x\ -6}$ and the **remainder is -3**

-3	1	6	3	-15
	↓	-3	-9	18
	1	3	-6	-3

Example:

Find the quotient and remainder when $2x^3 + 3x^2 - 5x + 3$ is divided by 2x + 1

Again we have to arrange the divisor into the form **x – h**

2x + 1 is the same as 2(x + ½) or **2(x – (-½))**

ignore the factor of 2 at the moment – we will deal with that separately.

-½	2	3	-5	3
	↓	-1	-1	3
	2	2	-6	6

The **quotient is** $\mathbf{2x^2 + 2x\ - 6}$ and the **remainder is 6**

However we now have to **divide the quotient by the factor of 2** that we took out.

So, the **quotient becomes:** $\mathbf{x^2 + x\ - 3}$ and the **remainder is 6**

NB we do NOT divide the remainder as well.

Unit 2 - 1.1 Polynomials

The Remainder Theorem

When any polynomial f(x) is divided by x – h the remainder is given by f(h)

We can find f(h) directly to obtain the remainder,

or we can use synthetic division.

The Factor Theorem

If the remainder when dividing a polynomial f(x) by x – h is 0 then x – h is a factor of f(x)

i.e. if f(h) = 0 then x – h is a factor.

This allows us to find factors of polynomials of any degree. Once we have a factor, we can divide by the factor using synthetic division, and obtain another polynomial of degree one less.

We can then repeat the process to obtain another factor, if one exists.

Using the Factor Theorem.

The easiest way to use the factor Theorem is as follows:

1. Look at the factors of the constant term in the polynomial f(x) these are the only possible values for h
2. Evaluate f(h) until you find f(h) = 0 and then you have a factor.
3. Once you have a factor, divide the polynomial by it using synthetic division and obtain the polynomial quotient which is of degree one less.
4. Repeat the process until you can find no more factors.

This is a follow on from the Remainder Theorem and is perhaps more important and certainly useful.

Example: Find the factors of : $f(x) = 2x^3 – 11x^2 + 17x – 6$

possible values for h are ±1, ±2, ±3, ±6,

Try h = 1
f(h) = f(1) = 2 – 11 + 17 – 6 = 2 this is not zero so (x – 1) is not a factor

Try h = -1
f(h) = f(-1) = -2 -11 -17 – 6 = -36 this is not zero so (x + 1) is not a factor

Try h = 2
f(h) = f(2) = 16 – 44 +34 – 6 = 0 so (x – 2) is a factor

Now obtain the quotient:

$$\begin{array}{r|rrrr} \mathbf{2} & \mathbf{2} & \mathbf{-11} & \mathbf{17} & \mathbf{-6} \\ & \downarrow & \mathbf{4} & \mathbf{-14} & \mathbf{6} \\ \hline & \mathbf{2} & \mathbf{-7} & \mathbf{3} & \multicolumn{1}{|r}{\mathbf{0}} \\ \cline{5-5} \end{array}$$

Quotient is: $2x^2 - 7x + 3$ so polynomial is **$(x – 2)(2x^2 - 7x + 3)$**

Now factorise the quadratic factor using two brackets:

$2x^2 - 7x + 3 \Rightarrow (2x -3)(x -3)$

Hence: **$f(x) = 2x^3 – 11x^2 +17x – 6$** factorises to **$f(x) = (x – 2) (2x -3)(x -3)$**

Solving Polynomial Equations

If h is a root of the equation f(x) = 0 then (x – h) is a factor of f(x) and so f(h) = 0

Recall the graph of f(x) a root is where f(x) crosses the x axis – in other words f(x) = 0

Consequently the value of x, at which f(x) = 0, is a root of the equation f(x) = 0

So if we can find h such that f(h) = 0 then we have a root of the equation f(x) = 0

Example: solve the equation $x^3 – 2x^2 – x + 2 = 0$

First find a factor – try possible values: ±1, ±2

f(1) = 1 – 2 – 1 + 2 ⇒ 0 so (x – 1) is a factor

Use synthetic division to divide f(x) by the factor

$$\begin{array}{r|rrrr} \mathbf{1} & \mathbf{1} & \mathbf{-2} & \mathbf{-1} & \mathbf{2} \\ & \downarrow & \mathbf{1} & \mathbf{-1} & \mathbf{-2} \\ \hline & \mathbf{1} & \mathbf{-1} & \mathbf{-2} & \multicolumn{1}{|r}{\mathbf{0}} \\ \cline{5-5} \end{array}$$

hence: $x^3 – 2x^2 – x + 2 = 0$ factorises to $(x – 1)(x^2 – x – 2) = 0$

now factorise the quadratic part to get: $(x – 1)(x – 2)(x + 1) = 0$

Hence solutions of the equation: $x^3 – 2x^2 – x + 2 = 0$

are: x = 1, x = 2 and x = -1

Unit 2 - 1.1 Polynomials

Finding approximate roots of the equation f(x) = 0

The previous method using the factor theorem will work providing the polynomial has factors

i.e. the roots are rational.

If they are not rational, the polynomial will not factorise and so we use a method to approximate the roots.

Solving by Iteration

Recall the graph of a function.

The roots are where f(x) crosses the x axis.

To one side of the root f(x) will be positive and on the other side of the root, f(x) will be negative.

So by finding two points such that f(x) is positive at one point and negative at the other, you know that the root must lie between the two points.

Take the middle point between these two points and depending upon whether this is positive or negative it will tell you on which side of the middle point the root lies.

Repeat this process until you have approached the root as close as the required accuracy.

If you want accuracy to 1 decimal place then you need to find the root with knowledge of the 2nd decimal place.

This is a process known as iteration.

Example:

Show that $x^3 - 3x + 1 = 0$ has a real root between 1 and 2

Find an approximation for the root to 1 decimal place.

$f(1) = 1 - 3 + 1 = -1$

$f(2) = 8 - 6 + 1 = 3$

$\therefore$ f(x) crosses the x axis between x=1 and x=2, indicating a root α there.

Now home in on the root – you may use your calculator here – **CAREFULLY!**

$f(1.5) \approx -0.13$ So $1.5 < \alpha < 2$

$f(1.7) \approx +0.81$ So $1.5 < \alpha < 1.7$

$f(1.6) \approx +0.30$ So $1.5 < \alpha < 21.6$

$f(1.55) \approx +0.07$ So $1.5 < \alpha < 1.55$

$f(1.54) \approx +0.03$ So $1.5 < \alpha < 1.54$

hence root is 1.5 correct to 1 decimal place

Unit 2 - 1.2 Quadratic Theory

Reminders:

$f(x) = ax^2 + bx + c$ $a \neq 0$ is a **quadratic function**

$3x^2 + 2x - 1$ is a **quadratic expression**
with $a = 3$, $b = 2$ and $c = -1$

$3x^2 + 2x - 1 = 0$ is a **quadratic equation**
(this one can be solved by factors)

A quadratic equation with roots 2 and -3 is
$(x - 2)(x + 3) = 0$ which multiplies out to $x^2 + x - 6 = 0$

Solving Quadratic Equations

We know the following methods for solving quadratic equations:

1. Graphically, where the graph crosses the x axis.
2. Factorise (put into 2 brackets)
3. Use the quadratic formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
4. Completing the square:

Example: Solve $x^2 - 2x - 4 = 0$ by completing the square

$(x - 1)^2 - 1 - 4 = 0$

$(x - 1)^2 - 5 = 0$

$(x - 1)^2 = 5$

$x = 1 \pm \sqrt{5}$ hence x = **3.24 or -1.24** (corr. to 2 d.p.)

Example: solve $3x^2 + 4x - 5 = 0$ using the formula

Here we have $a = 3$, $b = 4$ and $c = -5$

Use: $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

giving $x = \frac{-4 \pm \sqrt{4^2 - 4(3)(-5)}}{2(3)}$

Hence: $x = \frac{-4 \pm \sqrt{16 + 60}}{6}$ and $x = \frac{-4 \pm \sqrt{76}}{6}$

so **x = 0.79 or -2.12** (correct to 2 dec. pl)

The Discriminant

If we look at the formula for the solution of quadratic equations

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

we note that $b^2 - 4ac$ plays a fundamental role in determining the nature of the solutions.

we call $b^2 - 4ac$ the **Discriminant** – because it discriminates between different types of solution.

if $b^2 - 4ac > 0$ ⇒ two real and distinct roots

$b^2 - 4ac = 0$ ⇒ the roots are equal

$b^2 - 4ac < 0$ ⇒ there are no real roots.

Example:

For what value of p does the equation $x^2 - 2x + p = 0$ have equal roots.

$b^2 - 4ac = 4 - 4(1)(p) = 4 - 4p$ for equal roots this must be zero

so $4 - 4p = 0$ hence $4 = 4p$ and **p = 1**

Example:

Find the range of values for m for which $5x^2 - 3mx + 5 = 0$ has two real and distinct roots.

$b^2 - 4ac = 9m^2 - 4(5)(5) = 9m^2 - 100$

For real and distinct roots: $9m^2 - 100 > 0$ hence $9m^2 > 100$

$m^2 > \frac{100}{9}$ $m > +\frac{10}{3}$ $m < -\frac{10}{3}$

Example:

For what value of k does the graph $y = kx^2 - 3kx + 9$ touch the x-axis.

To touch the x axis, there must be equal roots so discriminant = 0

$b^2 - 4ac = 9k^2 - 4(k)(9) = 9k^2 - 36k$

$9k^2 - 36k = 0$ $9k(k - 4) = 0$ **so k = 0 or k = 4**

Unit 2 - 1.2 Quadratic Theory

Tangents to curves:

Example:

Find the value of c if the line $y = 5x + c$ is a tangent to the parabola $y = x^2 + 3x + 4$

See opposite for method:

The point of intersection is given by equation (……. 1) with $c = 3$

So $x^2 - 2x + 1 = 0$ which factorises to $(x – 1)(x – 1) = 0$ hence $x = 1$

when $x = 1$ the tangent equation gives us $y = 5x + 3 \Rightarrow y = 8$

So **point of intersection is (1, 8)**

To find the point of intersection of the line and parabola, solve the simultaneous equations:

$y = 5x + c$

$y = x^2 + 3x + 4$

by substitution we get $5x + c = x^2 + 3x + 4$

re-arranging gives: $x^2 - 2x + (4 – c) = 0$ ………. (1)

For the line to be tangent, the line must intersect the curve at **ONE** point only

i.e. we want equal roots so $b^2 - 4ac = 0$ hence $4 - 4(1)(4 - c) = 0$

and so $4 – 16 + 4c = 0$ $-12 + 4c = 0$ $c = 3$

So the **equation of the tangent will be $y = 5x + 3$**

Quadratic Inequalities

Solve this inequality $x^2 – 6x + 5 > 0$

First sketch the curve $y = x^2 – 6x + 5$

Factorising gives us $y = (x – 5)(x – 1)$

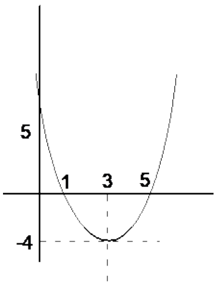
So curve crosses the x-axis at: **$x = 1$ and $x = 5$**

The y-intercept is **$y = 5$** (when $x = 0$)

Minimum of the quadratic lies on the line $x = 3$ (symmetry between $x = 1$ and $x = 5$)

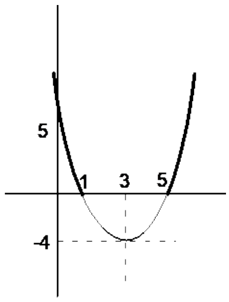
The minimum value is $y = 3^2 – 6(3) + 5 = 9 – 18 + 5 = -4$ **$y = -4$**

Sketch the curve, emphasise it where $y > 0$



(above) sketch of the graph

$y = x^2 – 6x + 5$



(above) part of the graph

where $y > 0$ i.e. $x^2 – 6x + 5 > 0$

hence the solution to the inequality $x^2 – 6x + 5 > 0$ is **$x < 1$ or $x > 5$**

Practical Example:

In the construction of an oil rig, the designers laid down these conditions for a rectangular helicopter landing pad.

(i) length to be 10m more than breadth

(ii) area of pad to lie between $375m^2$ and $600m^2$

Calculate the limits for the breadth of the pad.

See method of working opposite:

Summary:

- Form an inequality
- Sketch the graph using '=' signs
- Which part of graph is required
- Interpret the result

In summary with inequalities – sketch the curve and isolate the part that is relevant.

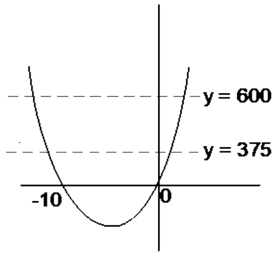
Let breadth of pad be x metres. So length of pad is x + 10 metres.

Area of pad is $x (x + 10)$ and area has to be between 375 m^2 and 600 m^2

So we have the inequality: $375 < x(x + 10) < 600$

Sketch the graph of $y = x(x + 10)$

We know that this graph crosses the x axis at $x = 0$ and $x = -10$



We need to find the values of x that correspond to $y = 600$ and $y = 375$ which will give us the limits for the breadth.

i.e. $y = 375$ and $y = x(x + 10) \Rightarrow y = x^2 + 10x$

So solve $x^2 + 10x – 375 = 0$

$(x + 25)(x - 15) = 0$

so $x = -25$ or $x = 15$ (discard negative value)

Now we need to solve $y = 600$ and $y = x^2 + 10x$

i.e. solve $x^2 + 10x - 600 = 0$

$(x + 30)(x – 20) = 0$ so $x = -30$ (discard) or $x = 20$

so at $x = 20$m the area will be 600 m^2 and at $x = 15$ area will be 375 m^2

hence **15 metres < breadth < 20 metres.**

Unit 2 - 2	**Integration**
Differential Equations An equation involving a derivative such as $\frac{dy}{dx} = 8x$ To solve this, we 'undo' the differentiation. This takes us back to $y = 4x^2$ + constant, which we write as $y = 4x^2 + c$ since any constant will differentiate to 0.	What we have done is to 'un-differentiate' and get back to the original function. $y = 4x^2 + c$ is called the anti-derivative of 8x
General Solution The general solution to $\frac{dy}{dx} = 8x$ is: $y = 4x^2 + c$ which represents a family of parabolas.	
Particular Solution To narrow it down to a particular parabola, we need more information (a boundary condition) such as when x = 1, y = 6. On substitution this gives us a value for c. Now we have the Particular Solution.	General solution: $y = 4x^2 + c$ But when x = 1, y = 6 so substitute to give: $6 = 4 + c$ So c = 2 **Particular Solution** is: $y = 4x^2 + 2$
Example: Find the particular solution of the differential equation $\frac{dy}{dx} = 8x - 1$ given by y = 5 when x = 1	The general solution is: $y = 4x^2 - x + c$ when y = 5, x = 1 so $5 = 4(1)^2 - (1) + c$ thus c = 2 The **particular solution is:** $\mathbf{y = 4x^2 - x + 2}$
Example: Kate and Mike make a simultaneous parachute jump. Their velocity after x seconds is $v = 5 + 10x$ m/s If they have fallen y metres then $v = \frac{dy}{dx} = 5 + 10x$ a) Find the distance y metres, they fall in x seconds, given y = 0 when x = 0 b) Calculate the distance they fall in 10 seconds.	If: $v = \frac{dy}{dx} = 5 + 10x$ then $y = 5x + 5x^2 + c$ (this is the general solution) Since we know that y = 0 when x = 0 then c = 0 (substitute in general soln) So the distance fallen in x seconds is given by: the Particular solution: $\mathbf{y = 5x + 5x^2}$ Hence the distance fallen in 10 seconds is given by: $y = 5(10) + 5(10)^2$ = **550 metres.**
Leibnitz' notation Leibnitz invented a useful notation for anti-derivatives: $\int 8x\,dx = 4x^2 + c$ In general $\int f(x)\,dx = F(x) + c$ which means effectively that $F'(x) = f(x)$ **The process of calculating the ant-derivative is known as Integration.**	The anti-derivative is called the integral and c is the constant of integration. F(x) is obtained from f(x) by integrating with respect to x Leibnitz notation is $\int f(x)\,dx$ The integral sign and the 'dx' cannot be separated – they are a pair, like a set of brackets.

Unit 2 - 2	**Integration**
Some useful rules $\int x^n dx = \frac{x^{n+1}}{n+1} + c \qquad n \neq -1$ $\int \left(f(x) + g(x) \right) dx = \int f(x)\,dx + \int g(x)\,dx$ $\int k\ f(x)\,dx = k\int f(x)\,dx$	INCREASE the index by 1, then divide by the new index. (note opposite of differentiation – which was multiply by the index, then DECREASE the index by 1) Integral of a sum is the sum of the integrals. A constant multiplier is carried along.
Examples: See opposite Treat each term separately and do not forget the constant of integration.	**Examples**: $\int x^3\ dx = \frac{x^4}{4} + c$ $\int 6x^2\ dx = \frac{6x^3}{3} + c = 2x^3 + c$ $\int x^2 + x\ dx = \frac{x^3}{3} + \frac{x^2}{2} + c$ $\int 3x^2 - 4\ dx \quad = \quad \frac{3x^3}{3} - 4x + c \quad = \quad x^3 - 4x + c$
Working with Gradients Given that the gradient of the curve y = f(x) is: $\frac{dy}{dx} = 3x^2 - 6x + 1$ and the point (3, 4) lies on the curve, find the equation of the curve. *(See opposite for solution).*	**Working:** Integrate giving $y = \int 3x^2 - 6x + 1\ dx$ so $y = \frac{3x^3}{3} - \frac{6x^2}{2} + x + c \ = \ x^3 - 3x^2 + x + c$ Now substitute the condition for the particular solution x = 3 and y = 4 to obtain c $4 = 3^3 - 3(3)^2 + 3 + c$ so $4 = 9 - 27 + 3 + c$ $4 = -15 + c$ $c = 19$ **Hence particular solution is : $y = x^3 - 3x^2 + x + 19$**
Fractional and Negative Indices Note that in order to integrate, you must have the function in straight line index form. **Example**: Integrate: $2 - \frac{1}{x^2} \quad \Rightarrow$ Integrate: $x - \frac{1}{\sqrt{x}} \quad \Rightarrow$ Integrate: $\left(u - \frac{1}{u} \right)^2 \quad \Rightarrow$	$\int 2 - x^{-2}\,dx \ \Rightarrow \ 2x - \frac{x^{-1}}{-1} + c \ \Rightarrow \ 2x + \frac{1}{x} + c$ $\int x - x^{-\frac{1}{2}}\,dx \ \Rightarrow \ \frac{x^2}{2} - \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + c \ \Rightarrow \ \frac{1}{2}x^2 - 2\sqrt{x} + c$ $\int u^2 - 2 + \frac{1}{u^2}\,du \Rightarrow \int u^2 - 2 + u^{-2}\,du \Rightarrow \frac{u^3}{3} - 2u + \frac{u^{-1}}{-1} + c$ $\Rightarrow \frac{1}{3}u^3 - 2u - \frac{1}{u} + c$

Unit 2 - 2

Integration

Example:

Integrate: $\frac{v^3+v}{v} \quad \Rightarrow$

$$\int \frac{v^3}{v}+\frac{v}{v}dv \;\Rightarrow\; \int v^2+1\,dv \;\Rightarrow\; \frac{v^3}{3}+v+c \;\Rightarrow\; \frac{1}{3}v^3+v+c$$

Applications:

The rate of growth per month (t) of the population P(t) of Carlos Town is given by the differential equation $\frac{dP}{dt}=5+8t^{\frac{1}{3}}$

a) Find the general solution of this equation.

b) Find the particular solution given that at present (t = 0), P = 5000

c) What will the population be 8 months from now ?

General solution given by:

$$\int 5+8t^{\frac{1}{3}}\,dt \quad = \quad 5t+\frac{8t^{\frac{4}{3}}}{\frac{4}{3}}+c, \qquad so \qquad P=5t+6t^{\frac{4}{3}}+c$$

To find c, put P=5000 and t = 0 so c = 5000

Hence $\mathrm{P}=5t+6t^{\frac{4}{3}}+5000$

8 months from now, substitute t = 8 into the equation

$\mathrm{P}=5(8)+6(8)^{\frac{4}{3}}+5000$ To deal with the $8^{\frac{4}{3}}$ recall that with fractional indices, the denominator specifies the root and the numerator the power.

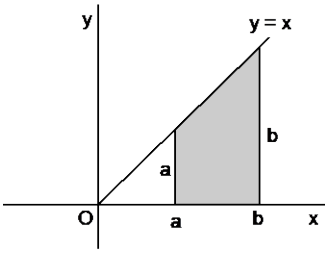
So, $8^{\frac{4}{3}} \Rightarrow \left(\sqrt[3]{8}\right)^4 \Rightarrow 2^4 \Rightarrow 16$

Hence the population after 8 months = 40 + 96 + 5000 = **5136**

The area under a curve

You have calculated many areas bounded by straight lines,

including rectangles, triangles and parallelograms.



For example the shaded area shown is:

$$\frac{1}{2}b^2-\frac{1}{2}a^2$$

(by considering the two squares of side a and b)

It is not so easy to calculate the area bounded by a curve.

We will work out a method for calculating the area bounded by the x-axis, the lines x = a and x = b and the curve y = f(x).

From the diagrams and working opposite, we can see that:

as h $\to$ 0 $\quad f(x) \le \lim\limits_{h\to 0}\frac{A(x+h)-A(x)}{h} \le f(x)$

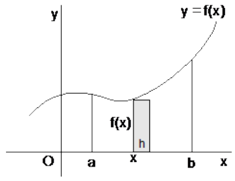
But $\quad \lim\limits_{h\to 0}\frac{A(x+h)-A(x)}{h} = A'(x)$

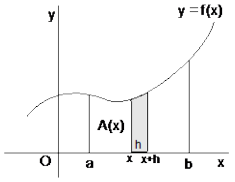
this is the definition of the derived function

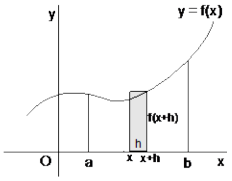
So f(x) = A'(x) and so $A(x)=\int f(x)\;dx$

by the definition of integration.

We will take A(x) to be the area under the curve up to x and starting at a and use this to find the area of a strip under the curve, h wide.







Area of shaded section	Area of shaded section	Area of shaded section
h × f(x)	A(x+h) – A(x)	h × f(x+h)
(simple rectangle)	Area up to (x+h) – Area up to x	(simple rectangle)

h × f(x) $\le$ A(x+h) – A(x) $\le$ h × f(x+h)

Dividing by h ($\ne$ 0) $\quad f(x) \le \frac{A(x+h)-A(x)}{h} \le f(x+h)$

Unit 2 - 2 Integration

Area under a curve – some notation

The area we wish to calculate, bounded by the x-axis, the lines x = a and x = b and the curve y = f(x)

is denoted by:

$$\int_a^b f(x)\ dx$$ we call this a **definite** integral

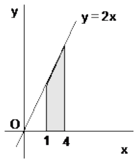
– read it as "the integral from a to b of f(x) dx"

In one sense, this integral represents the summing of all the strips of area under the curve from x = a to x = b, and in fact,

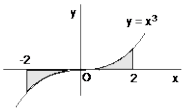
$\int$ is an elongated form of the letter S.

Example: show by shading in sketches, the areas associated with:

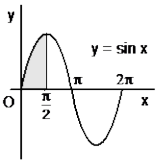
$$\int_1^4 2x\,dx$$



$$\int_{-2}^{2} x^3\ dx$$



$$\int_0^{\frac{\pi}{2}} \sin x\,dx$$

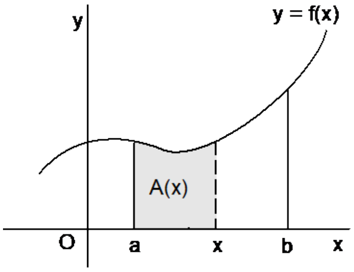


The Area under a curve – A formula; definite integrals

The area under the curve y = f(x) from x = a to x = b is

$$\int_a^b f(x)\,dx = \left[F(x)\right]_a^b = F(b) - F(a)$$

$\int_a^b f(x)\,dx$ is a definite integral with lower limit a and upper limit b.



A(x) is the area under the curve, starting at x = a
A(b) is the area under the curve from x = a to x = b, which is the area we wish to find.

From the last section $A(x) = \int f(x)\,dx$

Let $\int f(x)\,dx = F(x) + c$ where $F'(x) = f(x)$

Then $A(x) = F(x) + c$

and A(a) = 0 (from the diagram) $\Rightarrow$ 0 = F(a) + c so c = - F(a)

Now A(x) = F(x) – F(a)

So A(b) = F(b) – F(a) which is the area we are trying to find.

F(b) – F(a) is denoted by $\left[F(x)\right]_a^b$

Examples: Evaluate these integrals:

$$\int_1^3 x^3\ dx = \left[\frac{x^4}{4}\right]_1^3 = \frac{3^4}{4} - \frac{1^4}{4} = \frac{81}{4} - \frac{1}{4} = \frac{80}{4} = 20$$

$$\int_{-1}^{2} 2x(3x+1)\ dx = \int 6x^2 + 2x\ dx = \left[2x^3 + x^2\right]_{-1}^{2} = (16+4) - (-2+1) = 20 + 1 = 21$$

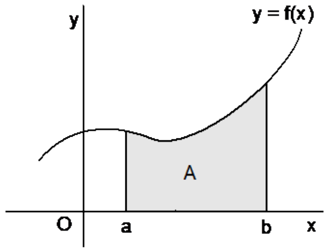
Unit 2 - 2 Integration

The Area under a curve – calculations

We use the fact that the area between the curve and the x-axis is given by:

$$\int_a^b f(x)\,dx = \left[F(x)\right]_a^b = F(b) - F(a)$$

where f(x) ≥ 0 and $a \le x \le b$



Example:

Calculate the area of the region bounded by:

the x-axis, the lines x = 1and x = 3 and the parabola $y = x^2 + 3$.

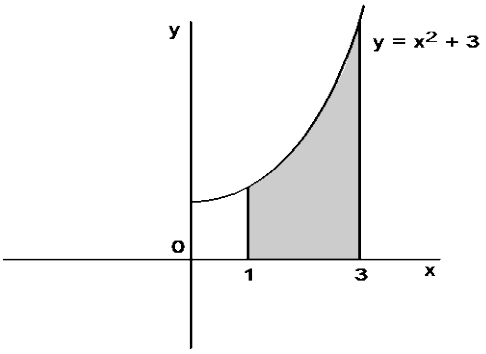
Solution:

Applying the above formula:

Required area = $\displaystyle\int_1^3 x^2 + 3\ dx$

$\Rightarrow \quad \Rightarrow \quad \left(\frac{27}{3}+9\right)-\left(\frac{1}{3}+3\right)$

$= 18 - 3\frac{1}{3} \quad = 14\frac{2}{3}$ units^2



Example:

Calculate the area of the region:

a) bounded by the x-axis, the line x = 2 and the parabola $y = x^2 - 1$ (A)

b) enclosed by the parabola and the x-axis (B)

Note:

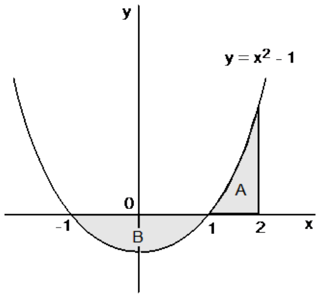
Integration will give **negative** values for **areas under the x-axis**, since for the shaded strip of width h, f(x) < 0 and h > 0 so h × f(x) is negative.

In the above example, the total area **cannot be found** by evaluating $\displaystyle\int_{-1}^{2} (x^2 - 1)\,dx$

Its value would be $1\frac{1}{3} + (-1\frac{1}{3}) = 0$

Areas for f(x) < 0 must be calculated separately, and the numerical values added together.

In the example above the total area of A + B would be $1\frac{1}{3} + 1\frac{1}{3} = 2\frac{2}{3}$ units^2



a) Area A =

$$\int_1^2 (x^2 - 1)\,dx = \left[\frac{x^3}{3} - x\right]_1^2 = \left(\frac{8}{3} - 2\right) - \left(\frac{1}{3} - 1\right) = 1\frac{1}{3}$$

b) Area B =

$$\int_{-1}^{1} (x^2 - 1)\,dx = \left[\frac{x^3}{3} - x\right]_{-1}^{1} = \left(\frac{1}{3} - 1\right) - \left(\frac{-1}{3} - (-1)\right) = -1\frac{1}{3}$$

This definite integral is **negative**, however the area has a **positive** value of $1\frac{1}{3}$

Unit 2 - 2 Integration

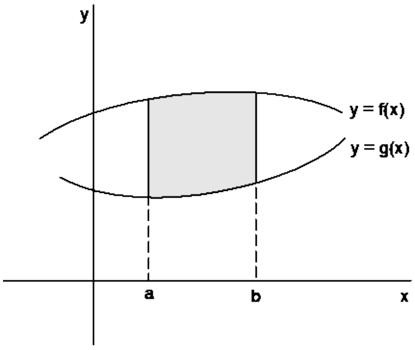
The area between two curves

The area between the curves y = f(x) and y = g(x) from x = a to x = b

= the area from the x-axis to the upper curve

– the area from the x-axis to the lower curve.

$$= \int_a^b \left(f(x) - g(x) \right) dx$$



Example:

Calculate the area enclosed by

the parabola $y = x^2$ and the line y = 2x.

$y = x^2$ meets y = 2x where

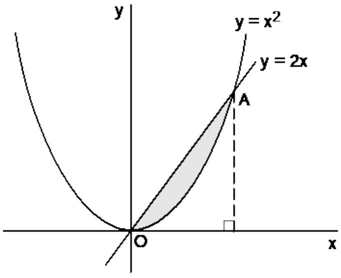
$x^2 = 2x$ i.e. x(x – 2) = 0

so x = 0 or x = 2

The area of the shaded region =

$$\int_0^2 2x\,dx - \int_0^2 x^2\,dx = \int_0^2 2x - x^2\ dx$$

$$= \left[x^2 - \frac{x^3}{3} \right]_0^2 \quad = \left(4 - \frac{8}{3} \right) - \left(0 - 0 \right) \quad = 1\frac{1}{3}$$

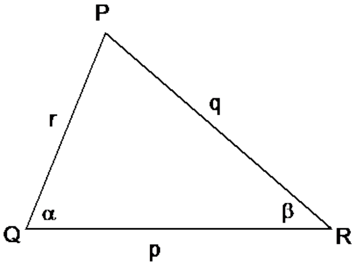


BEWARE:

Remember to watch out for areas below the x-axis. These will evaluate as **NEGATIVE**.

You must not integrate over a range which includes **POSITIVE AND NEGATIVE** values of f(x).

Unit 2 - 3.1 Calculations in 2 and 3 dimensions

Reminders: Basic Trigonometry	SOH-CAH-TOA $\text{Sin A} = \frac{\text{Opposite}}{\text{Hypotenuse}}$ $\text{Cos A} = \frac{\text{Adjacent}}{\text{Hypotenuse}}$ $\text{Tan A} = \frac{\text{Opposite}}{\text{Adjacent}}$
Sine Rule and Cosine Rule	Sine Rule $\frac{a}{\text{Sin A}} = \frac{b}{\text{Sin B}} = \frac{c}{\text{Sin C}}$ Cosine Rule $a^2 = b^2 + c^2 - 2bc\,\text{Cos A}$ $Cos\ A = \frac{b^2 + c^2 - a^2}{2bc}$
Area of Triangle	Area of Triangle Area of $\Delta\,\text{ABC} = \frac{1}{2}ab\,\text{Sin C}$
Related Angles – sketch them in on ASTC quadrants to convince yourself. sin (180° – A) = sin A sin (-A) = - sin A cos (180 – A) = - cos A cos (-A) = cos A	(the sine of an angle is the sine of its supplement - Recall ASTC)
Proofs: **Example**: a) Prove that the area of: $\Delta\text{PQR} = \frac{1}{2}q\,r\sin(\alpha+\beta)$ b) $p = \frac{q\sin(\alpha+\beta)}{\sin\alpha}$	
Method: a) Always start from what you know. Use formula for area of a triangle.	a) Area of $\Delta\,\text{ABC} = \frac{1}{2}ab\,\text{Sin C}$ So Area of $\Delta\,\text{PQR} = \frac{1}{2}q\,r\,\text{Sin P}$ but P = 180° - (α +β) and sin {180° - (α +β)} = sin (α +β) hence: Area of $\Delta\,\text{PQR} = \frac{1}{2}qr\,\text{Sin}\,(\alpha+\beta)$ **q.e.d.**
b) Looks like some variation on sine rule – so start with that.	b) Applying sine rule to ΔPQR gives: $\frac{p}{\text{Sin P}} = \frac{q}{\text{Sin Q}}$ However ∠ Q = α, so substitute and then re-arrange: $\frac{p}{\text{Sin P}} = \frac{q}{\text{Sin}\,\alpha} \Rightarrow p = \frac{q\ \sin P}{\text{Sin}\,\alpha}$ and from part a) we showed that sin P = sin (α +β) So: $p = \frac{q\ \sin\,(\alpha+\beta)}{\text{Sin}\,\alpha}$ **q.e.d.**

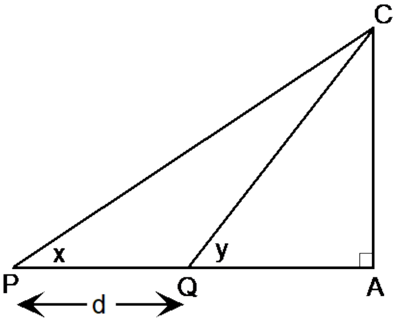
Calculations in 2 and 3 dimensions

Example:

Prove that:

a) $QC = \dfrac{d \sin x}{\sin(y - x)}$

b) $AC = \dfrac{d \sin x \sin y}{\sin(y - x)}$



Solution:

a) Start with sine rule: $\dfrac{\text{QC}}{\text{Sin x}} = \dfrac{\text{d}}{\text{Sin PCQ}}$

Now ∠PQC = 180° – y

so ∠PCQ = 180° – (x + (180° – y)) = 180 – (x + 180 – y)

= 180 – x –180 + y = y - x

hence: $\dfrac{\text{QC}}{\text{Sin x}} = \dfrac{\text{d}}{\text{Sin (y - x)}}$ then re-arrange to give:

$$\text{QC} = \frac{\text{d sin x}}{\text{Sin (y - x)}}$$ **q.e.d.**

b) We have QC from part a), we have angle y

we are trying to find AC – this is a right angled triangle

– which suggests SOH-CAH-TOA - the sine ratio.

So: $\sin y = \dfrac{\text{AC}}{\text{QC}} \Rightarrow \text{AC} = \text{QC} \sin y$

from previous part we have $\text{QC} = \dfrac{\text{d sin x}}{\text{Sin (y - x)}}$

so $\text{AC} = \text{QC sin y} = \dfrac{\text{d sin x sin y}}{\text{Sin (y - x)}}$ **q.e.d.**

Rules: Always start from something you know.

Look at what you are trying to prove

– does it look familiar in any way

– does it look similar to sine rule, cosine rule, SOH-CAH-TOA, area of triangle etc.

If it does then you know where to start.

Look at Left Hand Side of what you are trying to prove.

– Can you find a rule or formula linking it with something on the Right Hand Side.

Use the knowledge you have to get from the LHS to the RHS by substitution.

Unit 2 - 3.1 Calculations in 2 and 3 dimensions

Three Dimensions

We live in a 3 dimensional world – a world of length, breadth and height.

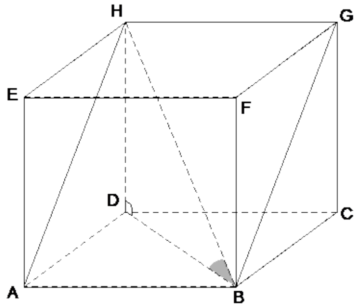
We can use the rules listed above, by applying them to 2-dimensional planes within the 3 dimensional solid.

(i) Angle between a line and a plane.

To find the angle between HB and the plane ABCDuse the perpendicular HD and form a right angled triangle Δ HDB

∠ HBD is the required angle

Calculations may involve Pythagoras and SOH-CAH-TOA



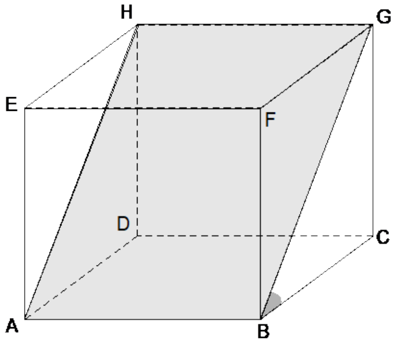
(ii) Angle between two planes.

To find the angle between planes ABGH and ABCD find their line of intersection AB.

Then a line in each plane perpendicular to AB, in this diagram, (BC and BG).

∠ CBG is the required angle.

∠ DAH would also do



Some terminology:

Face diagonal – this is a diagonal across a face. e.g. AH, ED, EG, FH etc.

Space diagonal – this is a diagonal linking two vertices which are not in the same face.

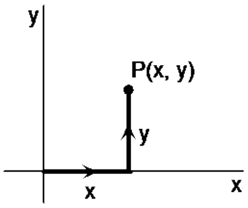
e.g. BH, AG, EC, DF

To find lengths of diagonals, calculations may involve Pythagoras and SOH-CAH-TOA.

Co-ordinates in 2 and 3 dimensions

To fix the position of a point on a plane (2 dimensions), you need two axes OX and OY.

P is the point (x, y)



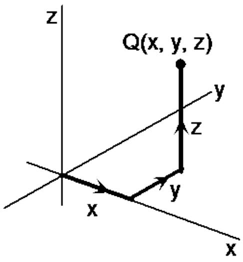
To fix the position of a point in space (3 dimensions), you need 3 axes – OX, OY and OZ.

and three co-ordinates – x, y and z

Q is the point (x, y, z)

In 3 dimensions, we usually show the

z direction vertically.



Unit 2 - 3.2 Compound Angle Formula

Reminders:

Related Angles: (Sketch the ASTC quadrants)

sin (180° – A) = sin A

(the sine of an angle is the sine of its supplement - Recall ASTC)

sin (90 – A) = cos A

sin (-A) = - sin A

cos (180 – A) = - cos A

cos (90 – A) = sin A

cos (-A) = cos A

Sin, cos, tan formulae

$$\frac{\sin A}{\cos A} = \tan A$$

$$\sin^2 A + \cos^2 A = 1$$

Radians and Degrees

π radians = 180°

Radians	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$
Degrees	30°	45°	60°
sin	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$
cos	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$
tan	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$

Compound Angle formulae

cos (A + B) = cos A cos B – sin A sin B

cos (A – B) = cos A cos B + sin A sin B

sin (A + B) = sin A cos B + cos B sin A

sin (A – B) = sin A cos B – cos B sin A

These formulae are true for all angles A and B whether in degrees or radians.

Double Angle Formulae

sin 2A = 2sin A cos A

cos 2A = $\cos^2$ A – $\sin^2$ A

cos 2A = 1 – $2\sin^2$ A

cos 2A = $2\cos^2$ A – 1

By re-arranging the formulae for cos 2A above we can also obtain:

$\cos^2$ A = ½ (1 + cos 2A)

$\sin^2$ A = ½ (1 – cos 2A)

These are all useful identities and are often used in proofs and calculations.

Put A = B in the above formulae for sin (A + B) and cos (A + B)

and by using $\sin^2 A + \cos^2 A = 1$ we can obtain the formula

for sin 2A and cos 2A

Rather than remember all the variations, try to remember the basic 4 identities

- sin (A + B), sin (A – B), cos (A + B), cos (A – B)

and how to derive the double angle formulae by putting A = B.

then by using $\sin^2 A + \cos^2 A = 1$

you can get from cos 2A to sin2 A or cos2 A

and then re-arrange to give $\cos^2 A$ *and* $\sin^2 A$

Unit 2 - 3.2 Compound Angle Formula

Examples: **1.** Using 75° = 30° + 45°, show that $\cos 75 = \frac{\sqrt{3}-1}{2\sqrt{2}}$ **2.** Using 3A = 2A + A prove that: sin 3A = 3 sin A – 4 sin^3 A	$\cos 75 \quad = \cos(30+45) \quad = \cos 30 \cos 45 - \sin 30 \sin 45$ $= \frac{\sqrt{3}}{2}\cdot\frac{1}{\sqrt{2}} - \frac{1}{2}\cdot\frac{1}{\sqrt{2}} \qquad = \frac{\sqrt{3}-1}{\sqrt{2}}$ $\sin 3A \quad = \sin(2A+A) \quad = \sin 2A \cos A + \sin A \cos 2A$ $\sin 2A = 2\sin A \cos A$ $\cos 2A = \cos^2 A - \sin^2 A$ $\cos^2 A = 1 - \sin^2 A$ $\therefore\ \sin 3A = 2\sin A \cos A \cos A + \sin A(1-2\sin^2 A)$ $\therefore\ \sin 3A = 2\sin A(1-\sin^2 A) + \sin A(1-2\sin^2 A)$ $\therefore\ \sin 3A = 2\sin A - 2\sin^3 A + \sin A - 2\sin^3 A$ $\therefore\ \sin 3A = 3\sin A - 4\sin^3 A$
Example 3. Express cos^4 x in the form a + b cos x + c cos 4x (Hint: start with cos^2 x = ½ (1 + cos 2x))	$\cos^2 x = \frac{1}{2}(1+\cos 2x)$ $\cos^4 x = \frac{1}{2}(1+\cos 2x)\cdot\frac{1}{2}(1+\cos 2x)$ $\cos^4 x = \frac{1}{4}(1+2\cos 2x+\cos^2 2x)$ $\cos 4x = \cos(2x+2x) = \cos^2 2x - \sin^2 2x$ $\sin^2 2x = 1-\cos^2 2x$ $\cos 4x = \cos^2 2x - 1 + \cos^2 2x$ $\cos 4x + 1 = 2\cos^2 2x$ $\frac{1}{2}(\cos 4x + 1) = \cos^2 2x$ $\cos^4 x = \frac{1}{4}\left(1+2\cos 2x+\frac{1}{2}(\cos 4x+1)\right)$ $\cos^4 x = \frac{1}{4}\left(1+2\cos 2x+\frac{1}{2}\cos 4x+\frac{1}{2}\right)$ $\cos^4 x = \frac{1}{4}+\frac{1}{2}\cos 2x+\frac{1}{8}\cos 4x+\frac{1}{8}$ $\cos^4 x = \frac{3}{8}+\frac{1}{2}\cos 2x+\frac{1}{8}\cos 4x$
Solving Trigonometric Equations **Example 4**: Solve cos 2x + cos x + 1 = 0 for 0 ≤ x ≤ 360° **Hence solutions are:** **x = 90°, 120°, 240°, 270°**	$\cos 2x + \cos x + 1 = 0$ $\cos^2 x - \sin^2 x + \cos x + 1 = 0$ $\cos^2 x - (1-\cos^2 x) + \cos x + 1 = 0$ $\cos^2 x - 1 + \cos^2 x + \cos x + 1 = 0$ $2\cos^2 x + \cos x = 0$ $\cos x(2\cos x + 1) = 0$ $\cos x = 0 \quad or \quad \cos x = -\frac{1}{2}$ $x = 90° \quad or \quad 270° \quad or \quad acute\ x = 60° \quad so\ \ x = 120°\ or\ 240°$

Unit 2 - 3.2	**Compound Angle Formula**
Example 5: Solve $\sin 2\theta + \cos \theta = 0$ for $0 \le x \le 2\pi$ **Hence solutions are:** $\theta = \frac{\pi}{2}, \frac{7\pi}{6}, \frac{3\pi}{2}, \frac{11\pi}{6}$	$\sin 2\theta + \cos\theta = 0$ $2\sin\theta\cos\theta + \cos\theta = 0$ $\cos\theta(2\sin\theta + 1) = 0$ $\cos\theta = 0 \quad or \quad \sin\theta = -\frac{1}{2}$ $\theta = \frac{\pi}{2}, \frac{3\pi}{2} \quad or \quad acute\ \theta = \frac{\pi}{6} \quad so\ \theta = \pi + \frac{\pi}{6}\ or\ 2\pi - \frac{\pi}{6}$
Example 6: Solve correct to 1 decimal place for $0 \le \theta \le 2\pi$ $5\cos 2\theta - \cos\theta + 2 = 0$ **Hence solutions are:** **θ = 0.9, 5.4 or 2.1 or 4.2 radians.** **Re-arranged in order of size:** **θ = 0.9, 2.1, 4.2 or 5.4 radians**	$5\cos 2\theta - \cos\theta + 2 = 0$ $5\left(\cos^2\theta - \sin^2\theta\right) - \cos\theta + 2 = 0$ $5\left(\cos^2\theta - \left(1 - \cos^2\theta\right)\right) - \cos\theta + 2 = 0$ $5\left(2\cos^2\theta - 1\right) - \cos\theta + 2 = 0$ $10\cos^2\theta - 5 - \cos\theta + 2 = 0$ $10\cos^2\theta - \cos\theta - 3 = 0$ $\left(5\cos\theta - 3\right)\left(2\cos\theta + 1\right) = 0$ $so\ \cos\theta = \frac{3}{5} \quad or \quad \cos\theta = -\frac{1}{2}$ $acute\ \theta = 0.927\,rad \quad or \quad acute\,\theta = \frac{\pi}{3}rad$ $hence\ \theta = 0.927\ or\ 2\pi - 0.927 \quad or \quad \theta = \pi - \frac{\pi}{3}\ or\ \pi + \frac{\pi}{3}\ rad$
Summary of methods Use double angle formula to expand: sin 2x or cos 2x Use $\sin^2 A + \cos^2 A = 1$ to switch from $\cos^2 x$ to $\sin^2 x$ or vice versa You will generally get a quadratic in cos x or sin x or a mixture. Factorise: i) common factor ii) two brackets Make sure you get ALL the roots.	

Unit 2 - 3.2 Compound Angle Formula

Graphs with equations

Recall sketching graphs of trigonometric functions

y = a sin nx   y = a cos nx

a = the amplitude

n = the number of cycles in 360° or 2π radians

the period of the function is: $\frac{360}{n}$ *or* $\frac{2\pi}{n}$

y = a sin (x + b)

the sine function is shifted b° to the left

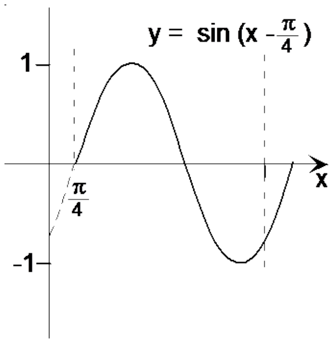
y = a cos (x - b)

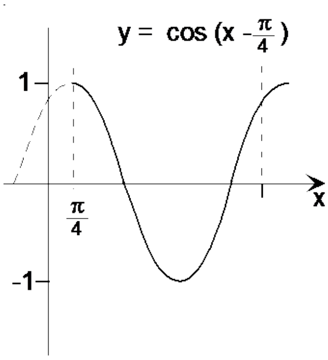
the cosine function is shifted b° to the right

equivalent statements can be made when working in radians.

Graphs of:

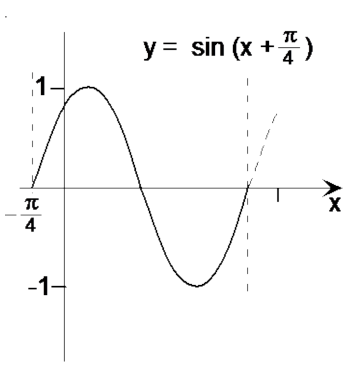
y = sin x and y = cos x moved $\frac{\pi}{4}$ to right.

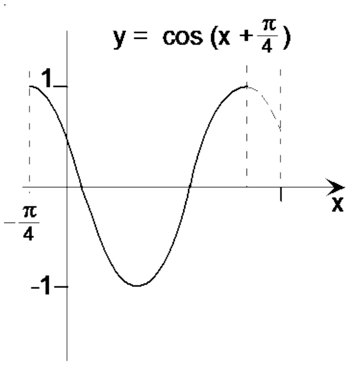




Graphs of:

y = sin x and y = cos x moved $\frac{\pi}{4}$ to left.





Unit 2 - 4 The Circle

The circle – centre O(0, 0) and radius r

$\mathbf{x^2 + y^2 = r^2}$

The equation of a circle is given by the locus of Point P

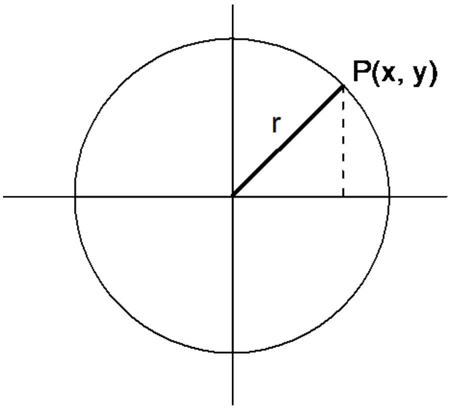
which describes a path at a constant distance r from the origin.

We need to find a relationship between x and y that satisfies this condition.

By Pythagoras: $x^2 + y^2 = r^2$

Hence the equation of the circle is:

$$x^2 + y^2 = r^2$$



Application:

Given the equation of a circle in the form

$x^2 + y^2 = r^2$

we can write down the radius.

Example: the radius of the circle: $x^2 + y^2 = 64$ is $\mathbf{r = 8}$

Example: the radius of the circle: $3x^2 + 3y^2 = 48$

first divide by 3 to get the form $x^2 + y^2 = r^2$

$x^2 + y^2 = 16$ so $\mathbf{r = 4}$

Application:

If we know that the circle is centred on the origin and passes through a given point, we can find its equation:

Example:

Find the equation of the circle centre O passing through P(3, 4)

using the distance formula, we can calculate OP as 5

This is the radius of the circle.

Hence $x^2 + y^2 = 25$

Application:

We can check that a point lies on a circle – if it does then it will satisfy the equation of the circle:

Example: Does the point R(12, -9) lie on the circle $x^2 + y^2 = 225$

LHS	RHS
$x^2 + y^2$	225
144 + 81	
225	

Since LHS = RHS, point R satisfies the equation, so R lies on the circle.

Alternative method:

If the point R(12, -9) lies on the circle, then OR will be equal to the radius of the circle (which is 15).

Using the distance formula we find that OR = 15, so R lies on the circle.

Example:

Find p if (p, 3) lies on the circle $x^2 + y^2 = 13$

(p, 3) must satisfy the equation of the circle, so:

$$p^2 + 3^2 = 13 \Rightarrow p^2 = 13 - 9 \Rightarrow p^2 = 4$$

$$\text{so } p = \pm 2$$

Example:

Does the point Q(7, -4) lie on the circle $x^2 + y^2 = 64$

The distance OQ (by the distance formula = $\sqrt{65}$)

This is larger than the radius of the circle,

so Q does NOT lie on the circle

Unit 2 - 4 The Circle

The circle centre C (a, b) and radius r

$(x – a)^2 + (y – b)^2 = r^2$

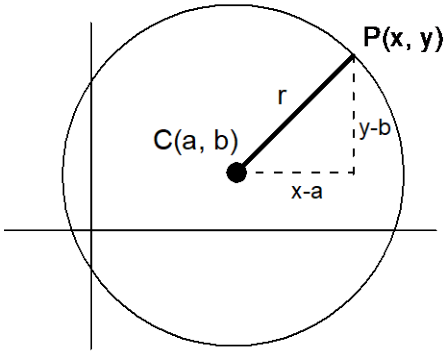
The equation of this circle is given by the locus of Point P which describes a path at a constant distance r from the centre, C(a, b)

We need to find a relationship between x and y that satisfies this condition.

By Pythagoras: $(x - a)^2 + (y – b)^2 = r^2$

Hence the equation of the circle is:

$$(x - a)^2 + (y – b)^2 = r^2$$



Applications:

Given the equation of a circle in the form

$$(x - a)^2 + (y – b)^2 = r^2$$

we can write down the co-ordinates of the centre of the circle and its radius.

Example:

State the centre and radius of the circle: $(x - 3)^2 + (y – 5)^2 = 9$

Circle is centred on (3, 5) and has a radius of 3

Example:

State the centre and radius of the circle: $(x + 7)^2 + (y – 2)^2 = 16$

Circle is centred on (-7, 2) and has a radius of 4

Example:

State the centre and radius of the circle: $x^2 + (y + 6)^2 = 49$

Circle is centred on (0, -6) and has a radius of 7

Example:

Write down the equation of the circle with centre (3, -1) and radius 2

Equation is: $(x - 3)^2 + (y + 1)^2 = 4$

Example:

Find the equation of the circle passing through the point P(3, -2) with centre C(-3, 0)

First we need to find the radius using the distance formula.

PC is the radius and PC = $\sqrt{(3-(-3))^2+(-2-0)^2} = \sqrt{36+4} = \sqrt{40}$

So $r^2 = 40$ Hence equation of circle is: $(x + 3)^2 + y^2 = 40$

Application:

If two circles touch, then we know that the distance between the centres of the two circles (using distance formula) is equal to the sums of their radii.

The converse also applies: If we want to find whether two circles touch, check the distance apart of the two centres and see if it is equal to the sum of their radii.

We can find the equation of a circle which passes through two points which form the diameter of the circle.

Example:

Find the equation of the circle passing through P(-1, -2) and Q(-1, 4) where PQ is a diameter.

If PQ is a diameter, then the mid point of PQ is the centre.

Half the length of PQ will be the radius (also distance from mid-point of PQ to P or Q)

Co-ordinates of Mid point of PQ (use mid-point formula) = M (-1 , 1)

Distance PQ = $\sqrt{(-1-(-1))^2+(-2-4)^2} = \sqrt{36}$ = 6 so radius = 3

Hence equation of circle is: : $(x + 1)^2 + (y - 1)^2 = 9$

However a quick sketch would save a lot of time and working !!!

Look at the co-ordinates of P(-1, -2) and Q(-1, 4) This is a vertical line on x = -1

You can immediately see that mid-point must be M(-1, 1) and length is clearly 6

Unit 2 - 4 The Circle

Applications and strategies:

You should recall previous work on the circle and be prepared to apply some of the facts you already know:

Summary:

Radius is half the diameter

Distance between centres of two circles which touch will be sum of radii.

Angle in a semi-circle is a right angle

A tangent to a circle is at right angles (90°) to the radius (or diameter)

Look for bisected chords (right angles to radius or diameter)

Look for symmetry

Look for isosceles triangles.

The shortest distance from a point to a line is a straight line perpendicular to the line.

Given three points P, Q, R, you can find the equation of the circle passing through them all.

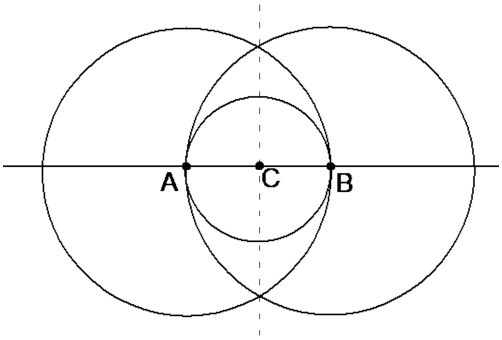
Join PQ (a chord) – Find gradient and mid-point. Find equation of perpendicular.

Join QR (a chord) – Find gradient and mid-point. Find equation of perpendicular.

Solve these two equations simultaneously – this gives co-ordinates of centre.

Distance from centre to P or Q or R will give radius.

Use radius and co-ordinates of centre to write down equation of the circle.



Example:

The small circle centre C has equation $(x + 2)^2 + (y + 1)^2 = 25$
The large circle, centres A and B, touch the small circle and AB is parallel to the x-axis.

Find: a) the centre and radius of each circle

b) the equations of the large circles.

Solution:

a)

For circle centred on C: co-ordinates of C are (-2, -1) and radius is 5

Hence: A is (-7, -1) and B is (3, -1) and radii of both circles are the same (= AB) which is the diameter of circle at C

radii of Circle A and circle B is 10

b)

Equation of circle centre A is: $(x + 7)^2 + (y + 1)^2 = 100$

Equation of circle centre B is: $(x - 3)^2 + (y + 1)^2 = 100$

Unit 2 - 4 The Circle

The general equation of a circle:

$x^2 + y^2 + 2gx + 2fy + c = 0$

with centre (-g, -f) and radius $\sqrt{g^2 + f^2 - c}$

provided that $g^2 + f^2 - c > 0$

Note that the coefficients of x^2 and y^2 must be 1.

The equation: $(x - 4)^2 + (y + 3)^2 = 4$

represents a circle with centre (4, -3) and radius 2

Multiplying out we get: $x^2 - 8x + 16 + y^2 + 6y + 9 = 4$

Re-arranging we get: $x^2 + y^2 - 8x + 6y + 21 = 0$

and this represents the same circle.

Can we show that: $x^2 + y^2 + 2gx + 2fy + c = 0$ represents a circle ?

Re-arranging we get: $x^2 + 2gx + y^2 + 2fy = -c$

Now complete the square $(x + g)^2 - g^2 + (y + f)^2 - f^2 = -c$

thus: $(x + g)^2 + (y + f)^2 = g^2 + f^2 - c$

we may choose to write this as:

$(x - (-g))^2 + (y - (-f))^2 = g^2 + f^2 - c$

and note that this represents a circle:

with centre (-g, -f) and radius $\sqrt{g^2 + f^2 - c}$

provided that $g^2 + f^2 - c > 0$

Strategies:

Given a circle in this form – we can find the centre and radius

We can then continue using strategies stated previously.

Example:

Show that the equation:

$3x^2 + 3y^2 - 12x + 24y - 36 = 0$

and find its centre and radius.

Divide throughout by 3 ⇒ $x^2 + y^2 - 4x + 8y - 12 = 0$

Compare with standard equation $x^2 + y^2 + 2gx + 2fy + c = 0$

this gives us: $2g = -4$ so **$g = -2$** and $2f = 8$ so **$f = 4$** and **$c = -12$**

Condition for a circle is: $g^2 + f^2 - c > 0$

and the coefficients x^2 and y^2 are equal to 1

$(-2)^2 + 4^2 - (-12) = 4 + 16 + 12 = 32$ which is > 0

so it is the equation of a circle.

The radius of the circle is $\sqrt{g^2 + f^2 - c}$ so **$r = \sqrt{32}$ (or $4\sqrt{2}$)**

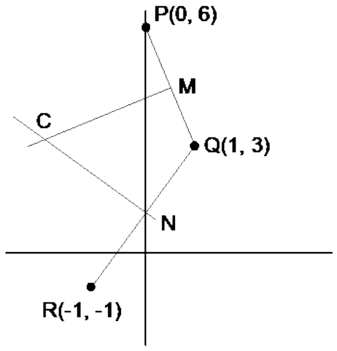
Centre is : (- g, - f) which gives **Centre = (2, -4)**

Example:

Find the equation of the circle through (-1, -1), (1, 3) and (0, 6)

A sketch is useful to apply labels.
Join PQ and QR.
Let mid-point of PQ be M and mid-point QR be N
Draw perpendiculars from M and N.

Where they meet at C is the centre of the circle.



Mid-point PQ is M(½ , 4½)

Gradient PQ is -3 Gradient MC = $\frac{1}{3}$

Equation of MC is: $\frac{y - 4\frac{1}{2}}{x - \frac{1}{2}} = \frac{1}{3} \Rightarrow 3y - \frac{27}{2} = x - \frac{1}{2}$

Re-arranging ⇒ $3y - x - 13 = 0$ …… (1) {**Equation MC**}

Repeat for NC Mid-point QR is N(0 , 1)

Gradient QR is 2 Gradient MC = $-\frac{1}{2}$

Equation of NC is: $\frac{y - 1}{x - 0} = -\frac{1}{2} \Rightarrow 2y - 2 + x = 0$

Re-arranging gives: $2y + x - 2 = 0$ …… (2) {**Equation NC**}

Solving (1) and (2) simultaneously we get: $x = -4$, $y = 3$

Hence centre of equation is C(-4, 3)

To find radius: find distance RC (or CQ or CP)

Using distance formula gives RC = 5

Hence equation of circle is: $(x + 4)^2 + (y - 3)^2 = 25$

or: $x^2 + 8x + 16 + y^2 - 6y + 9 = 25$ ⇒ **$x^2 + y^2 + 8x - 6y = 0$**

Unit 2 - 4 The Circle

Example:

a) Find the centres and radii of the circles:

$x^2 + y^2 = 4$

and $x^2 + y^2 – 8x + 6y + 24 = 0$

b) Sketch the circles and calculate the shortest distance between their circumferences.

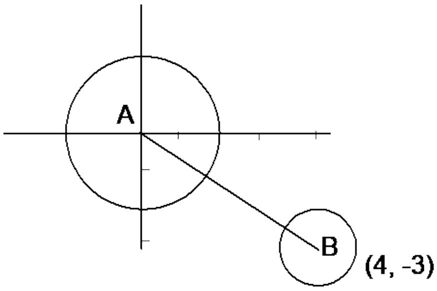
$x^2 + y^2 = 4$ **Centre (0, 0) and radius = 2**

$x^2 + y^2 – 8x + 6y + 24 = 0$ $2g = -8$ so $\mathbf{-g = 4}$ $2f = 6$ so $\mathbf{-f = -3}$

radius = $\sqrt{g^2 + f^2 - c}$ so **radius** = $\sqrt{}(16 + 9 – 24)$ **= 1**

Hence **Centre (4, -3) and radius 1**

sketch



The shortest distance between the circumferences will be along the line joining their centres.

Distance AB = 5 (distance formula)

Radius circle A = 2, radius circle B = 1

So distance between circumferences = 5 – 1 – 2 = 2

Distance between circumferences = 2the circles:

Tangents to a circle

To find the tangent to a circle at a given point P(x, y)

- Find the centre of the circle
- Find the gradient of the radius line from the centre to point P
- The tangent is perpendicular to the radius
- Find the gradient of the tangent
- Now find the equation of the tangent through point P

Example:

Find the equation of the tangent to the circle

$x^2 + y^2 – 4x + 6y – 12 = 0$ at the point P(5, 1)

Solution:

The centre C is (2, -3) { using centre at (–g, -f) }

Gradient PC = $\frac{1-(-3)}{5-2} \Rightarrow \frac{4}{3}$

Gradient of tangent = $-\frac{3}{4}$

Hence equation of tangent is: $y – 1 = -\frac{3}{4}(x – 5)$

$\Rightarrow$ $4y – 4 = -3x + 15$

Simplifying to : **Equation of tangent is:** $\mathbf{4y + 3x =19}$

Unit 2 - 4 | **The Circle**

Intersection of lines and circles

Use simultaneous equations to find the point of intersection.

Generally you will get two points of intersection.

Where the line enters and exits the circle,

tangency

unless the line is a tangent to the circle, in which case there will only be one point of intersection.

avoids the circle

or if the line misses the circle altogether, in which case there will be no points of intersection.

Example:

Find the co-ordinates of the points of intersection of

the line $5y – x + 7 = 0$

and the circle $x^2 + y^2 + 2x – 2y – 11 = 0$

Solution:

The lines meet when $5y – x + 7 = 0$ (1)

and $x^2 + y^2 + 2x – 2y – 11 = 0$ (2)

all we have to do is solve the equations simultaneously

Re-arrange (1) to give $x = 5y + 7$ and substitute into (2)

$\Rightarrow$ $(5y + 7)^2 + y^2 + 2(5y + 7) – 2y – 11 = 0$

$\Rightarrow$ $25y^2 + 70y + 49 + y^2 + 10y + 14 – 2y – 11 = 0$

$\Rightarrow$ $26y^2 +78y + 52 = 0$ (simplify by dividing by 26)

$\Rightarrow$ $y^2 +3y + 2 = 0$

$\Rightarrow$ $(y + 2)(y + 1) = 0$ hence $y = -2$ or $y = -1$

when $y = -2, x = -3$ and when $y = -1, x = 2$

So the points of intersection are: (-3, -2) and (2, -1)

Use of discriminant

We can also use the discriminant to give us information about the intersection of a line and a circle.

For example, by considering the quadratic equation which results from a simultaneous equation solution

We can deduce that:

Line meets the circle

in two distinct points real and distinct roots	$b^2 – 4ac > 0$
at one point only (tangent) equal roots	$b^2 – 4ac = 0$
at no point no real roots	$b^2 – 4ac < 0$

Example:

Find the values of k for $y = x + k$ to be a tangent

to the circle $x^2 + y^2 = 8$

Solution:

The line and circle intersect where

$$x^2 + (x + k)^2 = 8$$

(quadratic equation from simultaneous substitution)

i.e. $x^2 + x^2 + 2kx + k^2 = 8$

$\Rightarrow$ $2 x^2 + 2kx + k^2 – 8 = 0$

for a tangent we require only one solution i.e. equal roots

so $b^2 – 4ac = 0$

$\Rightarrow$ $4k^2 – 4(2)(k^2 – 8) = 0$

$\Rightarrow$ $-4k^2 – 64 = 0$ (divide throughout by 4)

$\Rightarrow$ $-k^2 – 16 = 0$ $\Rightarrow$ $k^2 = 16$

Hence $k = \pm 4$ (giving tangents $y = x \pm 4$)